Semiclassics with ’t Hooft flux and confinement of 4d gauge theories

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Introduction

Goal of this talk is to DERIVE

- Confinement of 4d SU(N) Yang-Mills theory, and
- Chiral effective Lagrangian for QCD,

in a CALCULABLE setup with SEMICLASSICAL methods.

However, confinement seems to be a strong-coupling physics.

Why can we expect that it's possible?

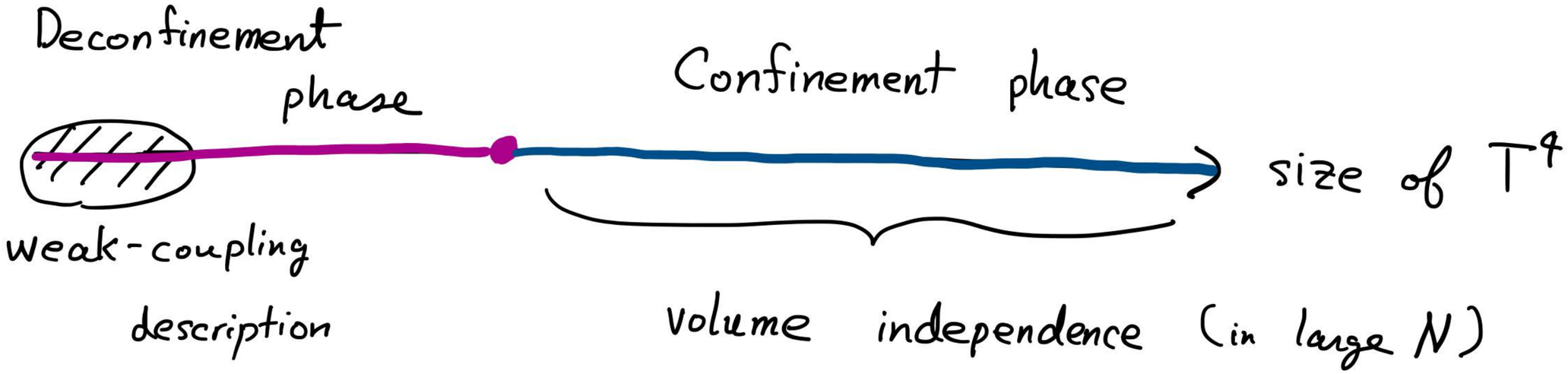
Volume independence

Eguchi, Kawai ('82): In the large-N limit,

$SU(N)$ YM theory on $\mathbb{R}^4$

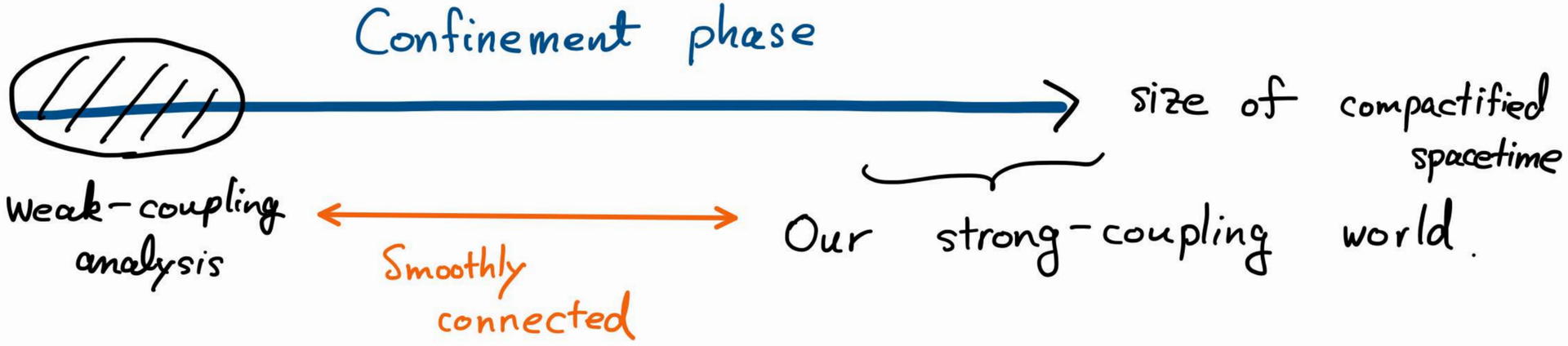
$=$ $SU(N)$ YM theory on T^4

as long as **the center symmetry is unbroken**.



Adiabatic Continuity to Weak-Couplings.

We'd like to achieve:



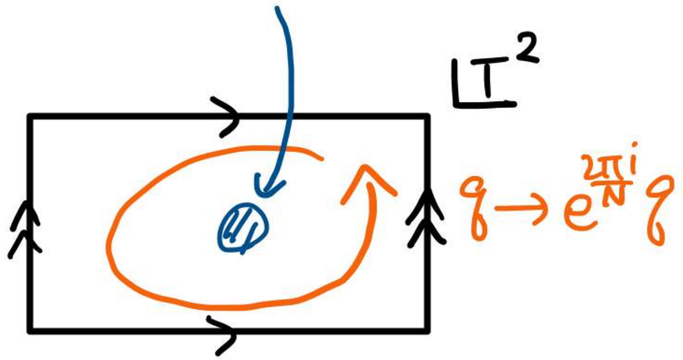
cf. · Twisted Eguchi-Kawai model (González-Arroyo, Okawa)

· Double-trace deformed YM
or YM theory w/ adjoint fermions
} on $\mathbb{R}^3 \times S^1$ (Ünsal, ... 2007–)

Conjecture (YT, Ünsal, '22)

YM, QCD on $\mathbb{R}^2 \times T^2$

w/ 't Hooft flux



Weak-coupling description

$\Longleftrightarrow$

Adiabatic Continuity

YM, QCD on $\mathbb{R}^4$

Strong-couplings

via Center Vortices

Supporting evidence

For small $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux,

we can use the Dilute Gas Approximation w/ center vortices.

It predicts

- (YM theory) $E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)
- ($\mathcal{N}=1$ SYM) $\langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$
- (QCD w/ non-commuting flavor twist $(N_c = N_f = N)$) $\langle \mathrm{tr}_{cf}(\bar{\psi})\, \mathrm{tr}_{cf}(\psi) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$

(Discrete chiral SSB)

- (QCD w/ $U(1)_B$ monopole flux) $S_{eff} \sim \int \left(|dU|^2 + \frac{1}{12\pi} \mathrm{tr}((U^\dagger dU)^3) + \chi_{top} (i \ln \det U - \theta)^2 \right)$

η' mass consistent with Witten-Veneziano formula

Plan

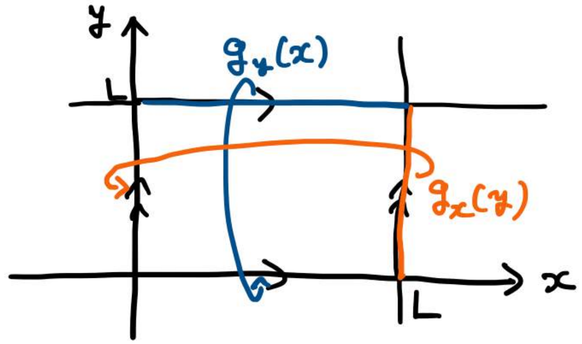
- Introduction
  - Confinement & adiabatic continuity
  - Our conjecture

- T^2 compactification with 't Hooft flux
  - Center vortex & confinement
  - Multi-branch structure of confinement vacua

YM theory

- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

't Hooft twisted boundary condition on $T^2 \ni (x, y)$

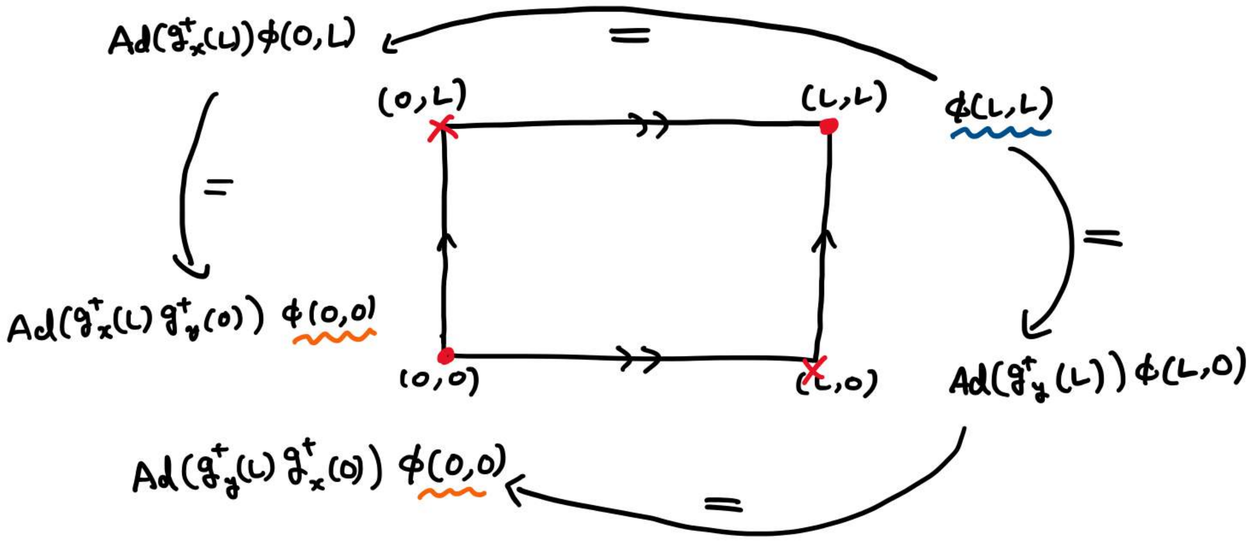


$$\begin{pmatrix} x \sim x+L \\ y \sim y+L \end{pmatrix}$$

$\phi(x, y)$: adjoint matter field

$$\phi(L, y) = \mathrm{Ad}\left(g_x^\dagger(y)\right) \phi(0, y)$$

$$\phi(x, L) = \mathrm{Ad}\left(g_y^\dagger(x)\right) \phi(x, 0)$$



Uniqueness of the matter wavefunction requires

$$g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{\frac{2\pi i}{N} n_{xy}}$$

↑ 't Hooft flux.

(cf When fundamental matters exist, the condition becomes

$$g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0) \quad . \quad)$$

Classical configuration and unbroken center symmetry

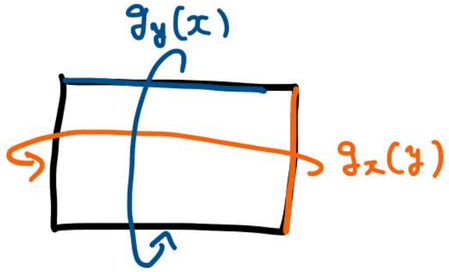
$F = 0$ gives the classical vacuum on T^2.

$\Rightarrow$ We can take a gauge so that $\boxed{a_\mu = 0}$.

However, Polyakov loops are forced to be nontrivial for the 't Hooft twist : $(n_{xy} = 1)$

$$g_x(y) \equiv S \propto \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 1 & & & 0 \end{pmatrix}$$

$$g_y(x) \equiv C \propto \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \quad (\omega = e^{\frac{2\pi i}{N}})$$



$(S, C \in SU(N)$ and $SC = CS\, e^{\frac{2\pi i}{N}})$.

$\Rightarrow P_x = S, \quad P_y = C.$

This is the $\boxed{\text{center-symmetric}}$ configuration. $(\mathrm{tr}(P_x) = 0$ etc.$)$

Perturbative analysis of SU(N) YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

- $\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry is unbroken.

- 2d gluons are gapped.

  $\Leftarrow$ Polyakov loops along T^2 are adjoint Higgs fields for $\mathbb{R}^2$.

  $P_3 = S$, $P_4 = C$ gives

  $$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N.$$

  Weak-coupling analysis is free from IR divergences.

- However, Wilson loops inside $\mathbb{R}^2$ obey perimeter laws.

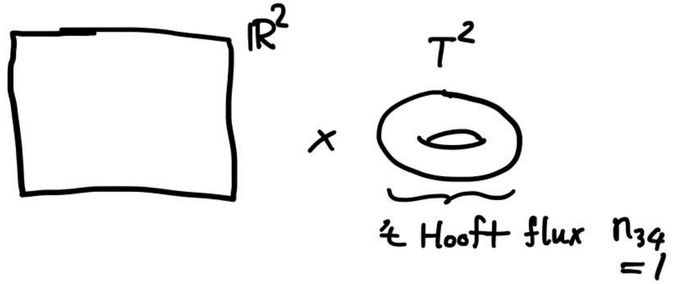
  $\Downarrow$

  We have to resolve this problem to achieve adiabatic continuity. $\Rightarrow$ Center vortex

Center vortex as a fractional instanton on $\mathbb{R}^2 \times T^2$

In this setup, the minimal topological charge is given by

$$Q_{top} = \frac{1}{8\pi^2}\int \mathrm{tr}(F\wedge F) = \frac{1}{N}$$



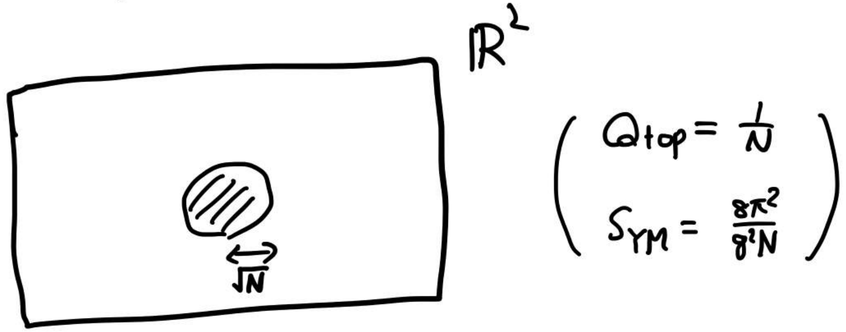
(More precisely, $Q_{top} \in \frac{1}{N}\left(\frac{-\epsilon_{\mu\nu\rho\sigma} n_{\mu\nu} n_{\rho\sigma}}{8}\right) + \mathbb{Z}$ (van Baal '82))

If there exists a self-dual configuration, its Yang-Mills action becomes

$$S_{YM} = \frac{8\pi^2}{g^2}|Q_{top}| = \frac{8\pi^2}{g^2 \cdot N}$$

González-Arroyo, Montero '98, Montero '99 numerically confirmed such a classical solution exists:

center vortex
or fractional instanton.



$$\left(\begin{array}{l} Q_{top} = \frac{1}{N} \\ S_{YM} = \frac{8\pi^2}{g^2 N} \end{array}\right)$$

(cf. García Pérez, González-Arroyo, '92, Itou '18)

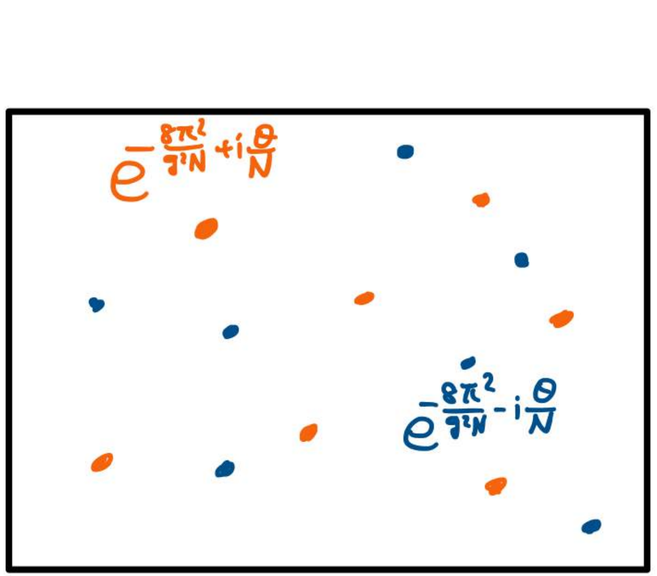
Dilute gas approximation

2d gluon fields are perturbatively gapped by 't Hooft twist.

$\Rightarrow$ Center vortex, or fractional instanton, does NOT have the size moduli.

$\Rightarrow$ Dilute gas approximation is available.

(* In 4d pure YM, DIGA is invalidated because of IR divergences.)



n : # of vortices

$\bar{n}$: # of anti-vortices

$$Q_{top} = \frac{n-\bar{n}}{N}.$$

Partition funcion on $M_2 \times T^2$ & θ-dependence

$M_2 \to \mathbb{R}^2$

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \left(V \cdot K e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}\right)^n \left(V \cdot K e^{-\frac{8\pi^2}{g^2 N} - i\frac{\theta}{N}}\right)^{\bar{n}}$$

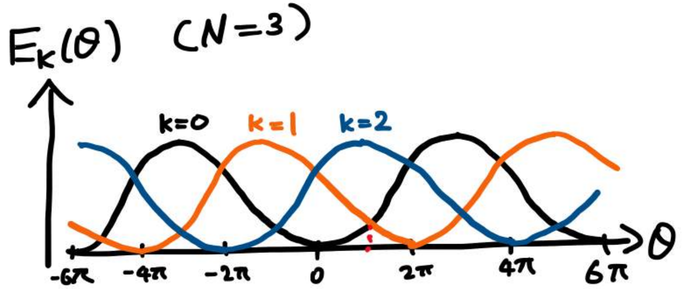
(vortex: $e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}$; anti-vortex: $e^{-\frac{8\pi^2}{g^2 N} - i\frac{\theta}{N}}$)

$$= \sum_{k=0}^{N-1} \exp\left[-V\left(-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\frac{\theta - 2\pi k}{N}\right)\right)\right]$$

$E_k(\theta)$: Ground-state energy densities

$\Rightarrow$

- N-branch structure of ground states.
- Each branch has a fractional θ-dependence.



Anomaly matching in 4d and 2d

4d SU(N) YM theory has an 't Hooft anomaly : (Gaiotto, Kapustin, Komargodski, Seiberg '17)

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B}\, Z_\theta[B]$$

$\Longrightarrow$ Confinement implies the multi-branch structure of vacua.

With T^2-compactification, $(\mathbb{Z}_N^{[1]})_{4d}$ splits into $(\mathbb{Z}_N^{[1]})_{2d} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

When 't Hooft flux n_{34} (mod N) is introduced, the 4d anomaly becomes

$$Z_{\theta+2\pi}[B_{2d}, A, A'] = e^{i\left(n_{34}\int B - \frac{N}{2\pi}\int A\wedge A'\right)}\, Z_\theta[B, A, A'].$$

$\Longrightarrow$ When $\gcd(n_{34}, N)=1$ (especially when $n_{34}=1$),

confinement implies the multi-branch structure. (cf. YT, Misumi, Sakai, '17. Yamazaki '17.)

Both properties are obtained by the center vortex.

Hamiltonian picture

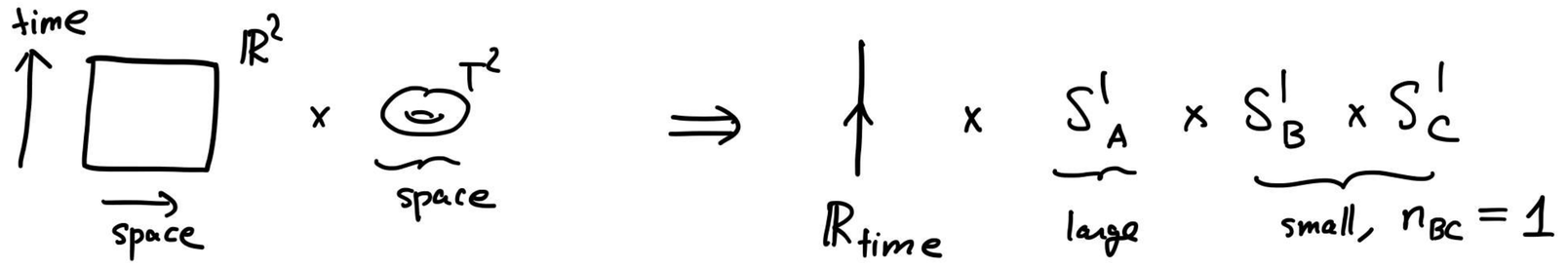
Using the dilute gas approximation of the YM path integral, we obtain

- N-branch structure of vacua
- Fractional θ-dependence $E_k(\theta) \sim \Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$
- Confinement of Wilson loops for non-trivial N-alities.

To get a better understandings,

let us construct the wave-function of these confining vacua.

Setup



(cf. Yamazaki, Yonekura '17 : $L_A, L_B \ll L_C$ ($\ll \Lambda^{-1}$) w/ $n_{BC} = 1$)

Perturbative vacua

We now have three holonomies (Polyakov loops along S^1_A, S^1_B, S^1_C): P_A, P_B, P_C.

't Hooft flux $n_{BC} = 1$: Classical vacuum $F_{BC} = 0$ gives

$$P_B = S \,, \qquad P_C = C \,.$$

We note that $n_{AB} = n_{AC} = 0$: Classical vacuum $F_{AB} = F_{AC} = 0$ gives

$$P_A P_B = P_B P_A \,, \qquad P_A P_C = P_C P_A \,.$$

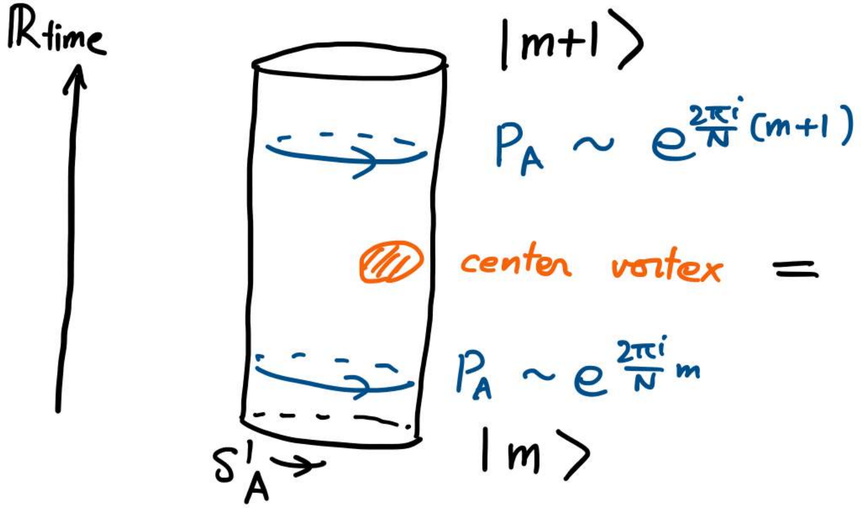
$$\Rightarrow \qquad P_A = e^{\frac{2\pi i}{N} m} \mathbb{1}_N \qquad (m = 0, 1, \cdots, N-1) \,.$$

Classically, $\mathbb{Z}_N$ center symmetry for P_A is spontaneously broken,

and we get N degenerate vacua.

$$|m\rangle \; : \; \frac{1}{N} \mathrm{tr}(P_A) \, |m\rangle = e^{\frac{2\pi i}{N} m} \, |m\rangle \,.$$

Tunneling amplitude via center vortex



= Tunneling from $|m\rangle$ to $|m+1\rangle$.

Within the one-center-vortex approximation,

$$\langle n | \exp(-T\hat{H}_{YM}) | m \rangle$$

$$\sim \underbrace{\delta_{nm}}_{\text{No tunneling}} + T \cdot L_A \cdot K e^{-\frac{8\pi^2}{g^2 N}} \left(\underbrace{e^{i\frac{\theta}{N}} \delta_{n,m+1}}_{\substack{\text{tunneling via} \\ \text{center vortex}}} + \underbrace{e^{-i\frac{\theta}{N}} \delta_{n,m-1}}_{\substack{\text{tunneling via} \\ \text{anti center vortex}}} \right)$$

Especially, $|m\rangle$ is not an eigenstate of $\hat{H}_{YM}$.

Eigenstate of $\exp(-T\hat{H}_{YM})$ and confining vacua

Let us consider the following wavefunction

$$|\widetilde{k}\rangle = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N} km} |m\rangle .$$

This is an eigenstate of $\exp(-T\hat{H}_{YM})$:

$$\exp(-T\hat{H}_{YM})\,|\widetilde{k}\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{n,m} e^{\frac{2\pi i}{N} km} |n\rangle \underbrace{\langle n| e^{-T\hat{H}_{YM}} |m\rangle}_{\sim\ \delta_{nm} + TL_A\, k e^{-\frac{8\pi^2}{g^2N}} \left(e^{i\frac{\theta}{N}} \delta_{n,m+1} + e^{-i\frac{\theta}{N}} \delta_{n,m-1}\right)}$$

$$\sim \exp\left[-T\cdot L_A \cdot \underbrace{\left(-2k e^{-\frac{8\pi^2}{g^2N}} \cos\left(\frac{\theta-2\pi k}{N}\right)\right)}_{= E_k(\theta)}\right] \cdot |\widetilde{k}\rangle .$$

$\Rightarrow |\widetilde{k}\rangle$ give the wavefunction of confining vacua for $\mathbb{R}^2 \times \underbrace{T^2}_{\text{'t Hooft flux}}$.

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  - Confinement & adiabatic continuity
  - Our conjecture
- T^2 compactification with 't Hooft flux
  - Center vortex & confinement
  - Multi-branch structure of confinement vacua
- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

QCD

't Hooft flux with fundamental quarks

We have seen that

Semiclassical theory w/ center vortices

$\Rightarrow$ Confinement vacua of YM theory.

Next, we would like to apply this idea for

Chiral effective Lagrangian of QCD.

Obstacle: We cannot introduce the 't Hooft flux when fundamental matters exist.

$U(1)_B$ vs $U(1)_q$

$$\text{Baryon numbers} = \frac{\text{quark numbers}}{N}$$

i.e. $$U(1)_B = U(1)_q / \mathbb{Z}_N .$$

Let us denote $U(1)_B$, $U(1)_q$ – gauge fields as A_B, A_q :

$$\frac{1}{N} A_B = A_q .$$

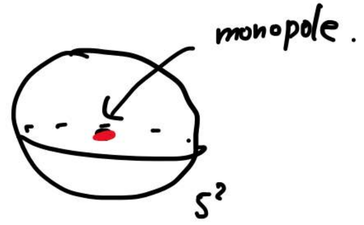
$$\left(\Leftarrow \quad A_{B\mu} J_B^\mu = A_{q\mu} J_q^\mu \quad \text{and} \quad J_B^\mu = \frac{1}{N} J_q^\mu \right)$$

However, this naive relation looks to be problematic when we consider monopoles :

$U(1)_B$ monopole : $\int_{S^2} dA_B = 2\pi .$ $\left(\Leftrightarrow \int_{S^2} dA_q = \frac{2\pi}{N} \right)$

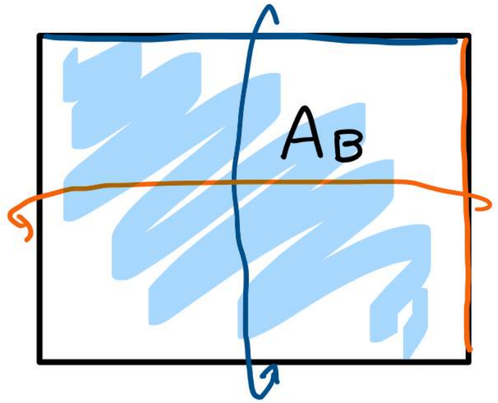
$U(1)_q$ monopole : $\int_{S^2} dA_q = 2\pi$

normalization doesn't match.



$U(1)_B$ monopole violates Dirac quantization.

$U(1)_B$ monopole flux & 't Hooft flux on T^2



quark field

$$\begin{cases} \psi(L,y) = g_x^\dagger(y)\, e^{-i\frac{\phi_x(y)}{N}}\, \psi(0,y) \\ \psi(x,L) = g_y^\dagger(x)\, e^{-i\frac{\phi_y(x)}{N}}\, \psi(x,0) \end{cases}$$

color-transition functions

$U(1)_B$-transition functions

Cocycle condition

$$g_x^\dagger(L)\, g_y^\dagger(0)\, e^{-i\frac{1}{N}(\phi_x(L)+\phi_y(0))} = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{-i\frac{1}{N}(\phi_y(L)+\phi_x(0))}$$

$U(1)_B$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_x(L)-\phi_x(0)) - (\phi_y(L)-\phi_y(0))$$

$$\Longrightarrow \quad g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0)\, \boxed{e^{\frac{2\pi i}{N}}}$$

't Hooft flux !!

(cf. YT, '18)

Perturbative spectrum

Gauge fields : Same with SU(N) YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

$\Rightarrow \mathbb{Z}_N$ gauge field.

Quark fields : Solve the Dirac zero-mode eq.

$$\left[\gamma_3\left(\partial_3 + i\frac{1}{N}A_{B,3}\right) + \gamma_4\left(\partial_4 + i\frac{1}{N}A_{B,4}\right)\right]\psi = 0.$$

$\Rightarrow$ For each 4d fundamental Dirac fermion,

there is a 2d massless Dirac fermion.

For N_f - flavor massless QCD, we have 2d N_f Dirac fermions.

$\longleftrightarrow$ non-Abelian bosonization $U(N_f)_1$ WZW model $\frac{1}{8\pi}\int_{M_2} \mathrm{tr}\,(dU^\dagger \wedge dU) + \frac{1}{12\pi}\int_{M_3} \mathrm{tr}\,[(U^\dagger dU)^3]$

w/ $U(N_f)$-valued field $U : M_2 \to U(N_f)$.

Center-vortex induced mass for η'

Let us construct the center-vortex vertex $\sim e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}$.

$U(1)$ axial anomaly requires the spurious symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}, \quad \theta \to \theta + 2N_f \alpha.$$

In the bosonized description

$$U \to e^{-2i\alpha} U.$$

$$\Rightarrow \quad e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}} \cdot (\det U)^{\frac{1}{N}} \quad \text{satisfies}$$

- $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry,
- $U(1)$ axial anomaly relation.

$$\Rightarrow \Delta S_{\text{center-vortex}} \sim \underbrace{\Lambda^2 (\Lambda L)^{\frac{5N-2N_f}{3N}}}_{\eta' \text{ mass}} \cdot \cos\left(\frac{i \ln(\det U) - \theta}{N}\right)$$

SUMMARY

YM, QCD on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux $\Longleftrightarrow$ YM, QCD on $\mathbb{R}^4$

Adiabatic Continuity

Strong-couplings

Semiclassics w/ center vortices

- (YM theory) $E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)
- ($\mathcal{N}=1$ SYM) $\langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$
- (QCD w/ non-commuting flavor twist ($N_c = N_f = N$)) $\langle \mathrm{tr}_{cf}(\bar{\psi})\, \mathrm{tr}_{cf}(\psi) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$

(Discrete chiral SSB)

- (QCD w/ $U(1)_B$ monopole flux) $S_{eff} \sim \int \left(|d U|^2 + \frac{1}{12\pi} \mathrm{tr}((U^\dagger dU)^3) + \chi_{top} (i \ln \det U - \theta)^2 \right)$

η' mass from Witten-Veneziano formula